

The Gibbons–Tsarev equation: symmetries, invariant solutions, and applications

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Abstract. In this paper we present the full classification of the symmetry-invariant solutions for the Gibbons–Tsarev equation. Then we use these solutions to construct explicit expressions for reductions of Benney’s moments equations, to get solutions of Pavlov’s equation, and to find integrable reductions of the Ferapontov–Huard–Zhang system, which describes implicit two-phase solutions of the dKP equation.

1. Introduction

The Gibbons–Tsarev equation

$$u_{yy} = (u_y + y) u_{xx} - u_x u_{xy} - 2 \quad (1)$$

has been widely known since it arose in [7] as a special case of a reduction of Benney’s moment equations, [4],

$$A_{n,t} + A_{n+1,x} + n A_{n-1} A_{0,x} = 0, \quad n \in \mathbb{N} \cup \{0\}. \quad (2)$$

Namely, suppose that A_2 and A_3 depend functionally on $p := A_0$ and $q := A_1$, that is, $A_2 = R(p, q)$, $A_3 = S(p, q)$ for some functions R and S . Then for all $n \geq 4$ moments A_n also depend functionally on A_0 and A_1 , $A_n = Q_n(p, q)$, where all the functions Q_n may be expressed recurrently in terms of R and S . Substituting for $A_2 = R(p, q)$, $A_3 = S(p, q)$ into (2) yields an over-determined system

$$\begin{cases} S_q &= R_p + R_q^2, \\ S_p &= R_q (R_p + p) - 2q, \end{cases} \quad (3)$$

which is compatible whenever $R_{pp} = (R_p + p) R_{qq} - R_q R_{pq} - 2$. This equation coincides with (1) after renaming $(q, p, R) \mapsto (x, y, u)$.

This origin of the Gibbons–Tsarev equation connects it directly with a model (also presented in the above-mentioned Benney’s work), which is meant to describe behavior of long waves on a shallow, inviscid and incompressible fluid. The Gibbons–Tsarev equation also arises in integrable models on algebraic curves, [12]. In [9, 10] the method of differential constraints was applied to find solutions of the Gibbons–Tsarev equation that are expressible in terms of solutions of Painlevé equations.

In this paper we use the methods of group analysis of differential equations, see, e.g., [13], to find solutions of the Gibbons–Tsarev equation that are invariant with respect to its symmetries. The research, as usually when Lie theory is applied, is performed in an algorithmic way, which involves reduction of the primary equation into equation in less independent variables than the primary one. The basic method includes arbitrary choice of symmetries, but one may confine this choice in some sense with the help of the adjoint representation of the symmetry algebra. In order to examine the problem of finding group-invariant solutions in as much systematic way as possible, we applied the method based on searching for an optimal system of one-dimensional subalgebras of the symmetry algebra of equation (1). Hence, every other group-invariant solution can be derived from one of the solutions we obtained.

As an immediate application of the invariant solutions of equation (1) we get explicit forms for four reductions of Benney’s moments equations. Two further applications are the following. First, as it was shown in [3], equation (1) arises as a symmetry reduction of equation [14, 5]

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}. \quad (4)$$

Thus, solutions to equation (1) provide solutions to equation (4). Second, the change of variables

$$z = u + \frac{1}{2} y^2 \quad (5)$$

transforms (1) to the first equation of the system

$$\begin{cases} z_{yy} &= z_y z_{xx} - z_x z_{xy} - 1, \\ w_{yy} &= z_y w_{xx} - z_x w_{xy}. \end{cases} \quad (6)$$

This system was shown in [6] to produce two-phase solutions for the dispersionless Kadomtsev–Petviashvili equation (dKP). Namely, if functions $P(r, s)$, $Q(r, s)$ satisfy

$$\begin{cases} P_{ss} &= P_s P_{rr} - P_r P_{rs} - 1, \\ Q_{ss} &= P_s Q_{rr} - P_r Q_{rs}, \end{cases} \quad (7)$$

then the system

$$\begin{cases} Q_r &= x + t(r + P_r), \\ Q_s &= y + tP_s, \end{cases} \quad (8)$$

implicitly defines a solution $r(t, x, y)$, $s(t, x, y)$ to the system

$$\begin{cases} r_t &= r r_x + s_y, \\ r_y &= s_x, \end{cases}$$

which is equivalent to the dKP equation

$$r_{yy} = r_{tx} - (r r_x)_x. \quad (9)$$

Each solution to (1) yields by substituting (5) into (6) a linear equation for w . We analyse symmetries of the obtained linear equations. Their corresponding reductions appear to be ordinary differential equations equivalent to Airy's equation,

$$v_{xx} = x v,$$

Weber's equation

$$v_{xx} = \left(\frac{1}{4} x^2 + \lambda \right) v,$$

Whittaker's equation

$$v_{xx} = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2} \right) v,$$

and Bessel's equation

$$v_{xx} = \left(\frac{1}{4} + \frac{4\mu^2 - 1}{4x^2} \right) v,$$

see, e.g., [18, 1]. While Airy's equation is not integrable in quadratures, [8], for Weber's equation, Whittaker's equation and Bessel's equation there exist an infinite number of values of the parameters λ , κ , μ such that those equations are integrable, see [17], [11] and [16]. Hence, we obtain an infinite number of cases when system (6) is integrable in quadratures. While the corresponding solutions to (7), (8) describe two-phase solutions for Eq. (9), their final form appears to be too complicated to write it explicitly.

2. The symmetry algebra of the Gibbons–Tsarev equation

Roughly speaking, a *symmetry group* of an equation

$$F(x, y, u, u_x, u_y, u_{xy}, u_{xx}, u_{yy}) = 0, \quad (10)$$

is a local group G of transformations g acting on some open subset of the space of independent and dependent variables $X \times Y \times U$, which transform solutions of the equation into solutions (for the strict definition see [13]). There is a one-to-one correspondence between symmetry group and its *infinitesimal generator*, which is a vector field of the form:

$$V = \xi_1(x, y, u) \frac{\partial}{\partial x} + \xi_2(x, y, u) \frac{\partial}{\partial y} + \eta(x, y, u) \frac{\partial}{\partial u}.$$

Every infinitesimal generator has its *characteristic function*, defined as $Q = \eta - \xi_1 u_x - \xi_2 u_y$, which is very useful from the computational point of view. In this paper, by symmetry we mean either a characteristic or a corresponding vector field, depending on a context. Finally, a *symmetry algebra* is a set of infinitesimal generators of symmetries, closed with respect to commutator $[\cdot, \cdot]$. For any two vector fields V_1, V_2 , their commutator is defined as $[V_1, V_2] := V_1 \circ V_2 - V_2 \circ V_1$.

If $u = f(x, y)$ is a solution of (10), then it is G -invariant if for any group transformation g the transformed function $(g \cdot f)(x, y)$ is a solution to Eq. (10), too. In what follows the notation will be widely based on the one adopted in [13].

2.1. Symmetry algebra

With the help of Jets software, [2], we found the symmetry algebra of equation (1), which is presented in the following table.

| Symmetry | Characteristic | Vector field |
|----------|--|---|
| ϕ_1 | $-y u_x + 2x$ | $y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial u}$ |
| ϕ_2 | $-x u_x - \frac{2}{3} y u_y + \frac{4}{3} u$ | $x \frac{\partial}{\partial x} + \frac{2}{3} y \frac{\partial}{\partial y} + \frac{4}{3} u \frac{\partial}{\partial u}$ |
| ϕ_3 | $-u_x$ | $\frac{\partial}{\partial x}$ |
| ϕ_4 | $-u_y - y$ | $\frac{\partial}{\partial y} - y \frac{\partial}{\partial u}$ |
| ϕ_5 | 1 | $\frac{\partial}{\partial u}$ |

The commutator table of this Lie algebra is the following:

| | ϕ_1 | ϕ_2 | ϕ_3 | ϕ_4 | ϕ_5 |
|----------|----------------------|---------------------|------------|----------------------|----------------------|
| ϕ_1 | 0 | $\frac{1}{3}\phi_1$ | $-2\phi_2$ | $-\phi_3$ | 0 |
| ϕ_2 | $-\frac{1}{3}\phi_1$ | 0 | $-\phi_3$ | $-\frac{2}{3}\phi_4$ | $-\frac{4}{3}\phi_5$ |
| ϕ_3 | $2\phi_2$ | ϕ_3 | 0 | 0 | 0 |
| ϕ_4 | ϕ_3 | $\frac{2}{3}\phi_4$ | 0 | 0 | 0 |
| ϕ_5 | 0 | $\frac{4}{3}\phi_5$ | 0 | 0 | 0 |

Note that ϕ_2 is a scaling symmetry, while ϕ_3 and ϕ_5 denote invariance of the set of solutions with respect to translations of x and u .

2.2. Adjoint representation

The full symmetry group of Gibbons–Tsarev equation is generated by five one-dimensional subgroups whose generators are presented in table (2.1). Reduction with respect to one of these 1-dimensional subgroups gives us an equation in $2 - 1 = 1$ variables. Any linear combination of symmetries is again a symmetry and it brings new reduction, which makes the task of finding all group-invariant solutions very tedious. However, it is easy to check, that if $f(x, y)$ is a G -invariant solution, then $(h \cdot f)(x, y)$ is $h G h^{-1}$ -invariant. This observation indicates the need of finding a set of solutions, which are invariant only to subgroups laying on separate orbits with respect to conjugation. As usual, we will operate on vector fields rather than subgroups of transformations themselves. The *adjoint representation* is defined as follows. For a given vector V from a Lie algebra denote by $Ad_{\epsilon V}$ a linear map on the Lie algebra, which is defined for every vector W from the Lie algebra as follows:

$$Ad_{\epsilon V} W := W - \epsilon [V, W] + \frac{\epsilon^2}{2!} [V, [V, W]] - \frac{\epsilon^3}{3!} [V, [V, [V, W]]] + \dots$$

The adjoint representation has a useful property of transforming vector W generating a subgroup G_W to the vector $Ad_{\epsilon V} W$ generating subgroup $h G_W h^{-1}$, where $h = \exp(\epsilon V)$. The adjoint representation for the symmetry algebra of the Gibbons–Tsarev equation is presented in the following table. The (i, j) -th entry is $Ad_{\epsilon \phi_i} \phi_j$.

| $Ad_{\epsilon \phi_i} \phi_j$ | ϕ_1 | ϕ_2 | ϕ_3 | ϕ_4 | ϕ_5 |
|-------------------------------|-----------------------------------|---------------------------------------|-----------------------------|--|-----------------------------------|
| ϕ_1 | ϕ_1 | $\phi_2 + \frac{\epsilon}{3} \phi_1$ | $\phi_3 - 2\epsilon \phi_5$ | $\phi_4 - \epsilon \phi_3 + \epsilon^2 \phi_5$ | ϕ_5 |
| ϕ_2 | $e^{-\frac{1}{2}\epsilon} \phi_1$ | ϕ_2 | $e^{-\epsilon} \phi_3$ | $e^{-\frac{2}{3}\epsilon} \phi_4$ | $e^{-\frac{4}{3}\epsilon} \phi_5$ |
| ϕ_3 | $\phi_1 + 2\epsilon \phi_5$ | $\phi_2 + \epsilon \phi_3$ | ϕ_3 | ϕ_4 | ϕ_5 |
| ϕ_4 | $\phi_1 + \epsilon \phi_3$ | $\phi_2 + \frac{2}{3}\epsilon \phi_4$ | ϕ_3 | ϕ_4 | ϕ_5 |
| ϕ_5 | ϕ_1 | $\phi_2 + \frac{4}{3}\epsilon \phi_5$ | ϕ_3 | ϕ_4 | ϕ_5 |

The following lemma presents an optimal system of one-dimensional subalgebras for the symmetry algebra of the Gibbons–Tsarev equation, by which a list of vectors generating conjugacy inequivalent one-parameter subgroups is meant, [13, § 3.3].

Lemma. *The optimal system of one-dimensional subalgebras consists of the subalgebras spanned by the following vectors: $\phi_1, \phi_2, \phi_3, \phi_4 + \alpha \phi_1, \phi_4 + \alpha \phi_5$, where α is an arbitrary constant.*

Proof is obtained by a standard computation, see, e.g., [13, § 3.3].

3. Reductions and invariant solutions

In this section we find solutions of equation (1) that are invariant with respect to the optimal system obtained in the above lemma. We use the method described, e.g., in [13, § 3.1].

3.1. Reduction with respect to ϕ_1

The ϕ_1 -invariant solutions of the Gibbons–Tsarev equation satisfy (1) and

$$\phi_1 = -y u_x + 2x = 0.$$

Solving the last equation for u_x and integrating gives $u = x^2 y^{-1} + W(y)$. Substituting this to (1) and solving for unknown function $W(y)$ yields

$$u = \frac{x^2}{y} + \beta y^3 + \gamma, \quad (11)$$

where β, γ are arbitrary constants.

3.2. Reduction with respect to ϕ_2

The ϕ_2 -invariant solutions of the Gibbons–Tsarev equation satisfy (1) and

$$\phi_2 = -x u_x - \frac{2}{3} y u_y + \frac{4}{3} u = 0.$$

Solving this we get $u = x^{4/3} v(\zeta)$ with $\zeta = y x^{-2/3}$. Inserting the outcome into (1) yields the ordinary differential equation

$$v_{\zeta\zeta} = \frac{2(\zeta v_{\zeta}^2 + (2v + 3\zeta^2)v_{\zeta} - 2\zeta v + 9)}{8\zeta v + 4\zeta^3 - 9}. \quad (12)$$

The point symmetries of this equation are trivial, so the methods of group analysis can not be applied to its integration. The general solution to (12) may be extracted from results of [15]. It has the parametric form

$$\begin{cases} v &= -\frac{3^{2/3}}{2(1 + \epsilon_1 + \epsilon_2)^{1/3}} \cdot \frac{P_2(t)}{(P_3(t))^{2/3}}, \\ \zeta &= -\frac{3^{4/3}}{2(1 + \epsilon_1 + \epsilon_2)^{2/3}} \cdot \frac{P_4(t)}{(P_3(t))^{4/3}} \end{cases} \quad (13)$$

with

$$\begin{aligned} P_2(t) &= (\epsilon_1 + \epsilon_2 t)^2 + \epsilon_1 + \epsilon_2 t^2, \\ P_3(t) &= (\epsilon_1 + \epsilon_2 t)^3 - \epsilon_1 - \epsilon_2 t^3, \\ P_4(t) &= (1 + 2(\epsilon_1 + \epsilon_2))(\epsilon_1 + \epsilon_2 t)^4 + \epsilon_2(2(1 + \epsilon_1 - \epsilon_2^2) + \epsilon_2)t^4 - 4\epsilon_1\epsilon_2^2 t^3 \\ &\quad - 2\epsilon_1\epsilon_2(1 + \epsilon_1 + \epsilon_2)t^2 - 4\epsilon_1^2\epsilon_2 t + \epsilon_1(\epsilon_1 + 2(1 + \epsilon_2 - \epsilon_1^2)), \end{aligned} \quad (14)$$

where ϵ_1 and ϵ_2 are arbitrary constants and t is a parameter. Since

$$\det \begin{pmatrix} \frac{\partial v}{\partial \epsilon_1} & \frac{\partial v}{\partial \epsilon_2} \\ \frac{\partial v_{\zeta}}{\partial \epsilon_1} & \frac{\partial v_{\zeta}}{\partial \epsilon_2} \end{pmatrix} = \frac{3^{5/3}(1 + \epsilon_1 + \epsilon_2)^{2/3}(t-1)^2(\epsilon_1 + (1 + \epsilon_2)t)^2(1 + \epsilon_1 + \epsilon_2 t)^2}{8(P_3(t))^{8/3}} \neq 0,$$

system (13), (14) indeed defines the general solution to Eq. (12). This fact was not proved in [15]. This solution is very complicated, so we will not use it in the constructions of Section 4.

3.3. Reduction with respect to ϕ_3

For ϕ_3 -invariant solutions of the Gibbons–Tsarev equation we have

$$\phi_3 = -u_x = 0,$$

so they do not depend on x and thus satisfy $u_{yy} = -2$. Hence these solutions are of the form

$$u = -y^2 + \beta y + \gamma \quad (15)$$

with $\beta, \gamma = \text{const.}$

3.4. Reduction with respect to $\phi_4 + \alpha\phi_1$

Solutions of the Gibbons–Tsarev equation that are invariant w.r.t. $\phi_4 + \alpha\phi_1$ satisfy (1) and

$$\phi_4 + \alpha\phi_1 = -\alpha y u_x - u_y - y + 2\alpha x = 0.$$

When $\alpha \neq 0$, we solve this equation for u_x , substitute the output into (1) and obtain the reduced equation

$$u_{yy} = \frac{2(x - \alpha y^2)}{y(2x - \alpha y^2)} u_y - \frac{4\alpha x^2}{y(2x - \alpha y^2)}.$$

This is a linear ordinary differential equation with x treated as a parameter. Solutions of this equation are of the form

$$u = 2\alpha x y - \frac{2}{3}\alpha^2 y^3 + W_1(x) \cdot |2x - \alpha y^2|^{\frac{3}{2}} + W_2(x),$$

where $W_1(x)$ and $W_2(x)$ are arbitrary (smooth) functions of x . By substituting this solution to (1) we obtain that $W_1(x) = \beta = \text{const}$ and $W_2(x) = -\alpha^{-1}x + \gamma$, $\gamma = \text{const}$. Finally, solution invariant with respect to symmetry $\phi_4 + \alpha\phi_1$ is of the form:

$$u = (2\alpha y - \alpha^{-1})x - \frac{2}{3}\alpha^2 y^3 + \beta |\alpha y^2 - 2x|^{\frac{3}{2}} + \gamma. \quad (16)$$

When $\alpha = 0$, we have $u_y = -y$, so $u = -\frac{1}{2}y^2 + W_1(x)$. But substituting for this into (1) gives a contradiction.

3.5. Reduction with respect to $\phi_4 + \alpha\phi_5$

Solutions of the Gibbons–Tsarev equation that are invariant w.r.t. $\phi_5 + \beta\phi_4$ satisfy (1) and

$$\phi_4 + \alpha\phi_5 = -u_y - y + \alpha = 0.$$

This gives $u = -\frac{1}{2}y^2 + \alpha y + W(x)$. Substituting to (1) and solving for $W(x)$ gives the solution of the form

$$u = \frac{1}{2\alpha}x^2 - \frac{1}{2}y^2 + \alpha y + \beta x + \gamma \quad (17)$$

with $\beta, \gamma = \text{const.}$

4. Applications

4.1. Reductions of Benney’s moments equation

Renaming $(q, p, R) \mapsto (x, y, u)$ in system (3) and substituting for a solution of equation (1) into the resulting system

$$\begin{cases} S_x &= u_y + u_x^2, \\ S_y &= u_x(u_y + y) - 2x, \end{cases}$$

we obtain a compatible system for S . This system has the following solutions that correspond to the invariant solutions (11), (15), (16), (17) of the Gibbons–Tsarev equation, respectively:

$$S = \frac{x^3}{y^2} + 3\beta xy^2 + \delta,$$

$$S = (\beta - 2y)x + \delta,$$

$$S = \frac{\beta(3\alpha^2 y - 2)}{\alpha} |\alpha y^2 - 2x|^{\frac{3}{2}} - \frac{1}{4}\alpha^2(4\alpha + 9\beta^2)y^4 - 4xy + \frac{1}{\alpha^2}x \\ + \frac{4}{3}\alpha y^3 + (\alpha - 9\beta^2)x^2 + \frac{1}{2\alpha}(18\alpha^2\beta^2x + 4\alpha^3x - 1)y^2 + \delta,$$

$$S = \frac{x^3}{3\alpha^2} + \frac{\beta x^2}{\alpha} + (\alpha + \beta^2 - y)x + \alpha\beta y + \delta,$$

where δ is an arbitrary constant.

4.2. Solutions to equation (4)

Equation (4) has solutions of the form

$$u(t, x, y) = v(\tau, y) - 2tx - t^2y, \quad (18)$$

where $\tau = x + ty$ and function $v(\tau, y)$ is a solution of equation (1) with x replaced by τ . Since we know four explicit solutions (11), (15), (16), (17) of the Gibbons–Tsarev equation, after substituting them into (18) we obtain four explicit solutions of equation (4). They are, respectively,

$$u = \frac{x^2}{y} + \beta y^3 + \gamma,$$

$$u = -y^2 + \beta y - 2tx - t^2y + \gamma,$$

$$u = (2\alpha y - 2t - \alpha^{-1})(x + ty) - \frac{2}{3}\alpha^2 y^3 + \beta |2x - \alpha y^2 + ty|^{\frac{3}{2}} + t^2y + \gamma,$$

$$u = \frac{1}{2\alpha}(x + ty)^2 - \frac{1}{2}y^2 + \alpha y + \beta(x + ty) - 2tx - t^2y + \gamma.$$

4.3. Reductions of the Ferapontov–Huard–Zhang system

In this section we study solutions of system (6) that correspond to the obtained solutions (11), (15), (16), (17) of the Gibbons–Tsarev equation. Each solution of (1) yields by substituting (5) into (6) a linear equation for w . Any linear equation admits trivial symmetries, that is, symmetries of the form $w_0 \frac{\partial}{\partial w}$, where w_0 is a (fixed) arbitrary solution of the equation. We consider nontrivial symmetries of the obtained linear equations. These symmetries allow one to reduce their equations to ordinary differential equations. For each one of these ODEs we indicate all the cases when the ODE is integrable in quadratures.

4.3.1. Solution (11)

For solution (11) the second equation of system (6) takes the form

$$w_{yy} = (3\beta y^2 + y - x^2 y^{-2}) w_{xx} - 2xy^{-1} w_{xy}.$$

After the change of variables $x = \tilde{x} \tilde{y}$, $y = \tilde{y}$, $w = \tilde{w}$ and dropping tildes the last equation acquires the form $w_{yy} = (3\beta y + 1) y^{-1} w_{xx}$. This equation has a nontrivial symmetry $w_x - \lambda w$, where λ is an arbitrary constant. The corresponding reduction $w = e^{\lambda x} v(y)$ gives an ODE $v_{yy} = \lambda^2 (3\beta y + 1) y^{-1} v$. After the scaling $\tilde{y} = 2\sqrt{3}\lambda\beta^{1/2} y$ and dropping tildes we have Whittaker's

$$v_{yy} = \left(\frac{1}{4} - \frac{\kappa}{y} \right) v \tag{19}$$

with $\kappa = -\frac{1}{6}\sqrt{3}\lambda\beta^{1/2}$. From results of [16] it follows that equation (19) is integrable in quadratures whenever $\kappa \in \mathbb{Z}$. Therefore for each choice of β there exists an infinite number of values for λ such that equation (19) is integrable in quadratures.

4.3.2. Solution (15)

Without loss of generality it is possible to put $\beta = 0$ in solution (15). Then we have

$$w_{yy} = -y w_{xx}. \tag{20}$$

The nontrivial symmetries of this equation are the following: $\psi_1 = w_x - \lambda w$ with $\lambda = \text{const}$, $\psi_2 = 3xw_x + 2yw_y$, and $\psi_3 = 12xyu_y + 3xw - (4y^3 - 9x^2)u_x$.

The ψ_1 -invariant solution of equation (20) is of the form $w = e^{\lambda x} v(y)$, where v satisfies $v_{yy} = -\lambda^2 y v$. After rescaling $y = -\lambda^{2/3} \tilde{y}$ and dropping tildes the last equation acquires the form of Airy's equation

$$v_{yy} = y v,$$

which is not integrable in quadratures, [8].

The ψ_2 -invariant solution of equation (20) is of the form $w = v(\eta)$ with $\eta = xy^{-3/2}$, where v is a solution of equation

$$v_{\eta\eta} = -\frac{10\eta^2}{4\eta^3 + 9} v_\eta.$$

The general solution of this equation is

$$U = c_1 + c_2 \int \frac{d\eta}{(4\eta^3 + 9)^{5/6}}.$$

The last integral can not be expressed in elementary functions, [17].

For a ψ_3 -invariant solution we have

$$w = \frac{y}{(4y^3 + 9x^2)^{5/6}} v(\sigma), \quad \sigma = \frac{4y^3 + 9x^2}{y^{3/2}},$$

where $v(\sigma)$ is a solution to $v_{\sigma\sigma} = -3^{-1}\sigma^{-1}v_\sigma$. This equation is integrable in quadratures, its general solution reads $v = c_1 + c_2\sigma^{2/3}$, where $c_1, c_2 = \text{const.}$ Therefore we have

$$w = \frac{c_1 y}{(4y^3 + 9x^2)^{5/6}} + \frac{c_2}{(4y^3 + 9x^2)^{1/6}}.$$

4.3.3. Solution (16) with $\beta \neq 0$

For solution (16) in the case of $\beta \neq 0$ the second equation of system (6) acquires the form

$$\begin{aligned} w_{yy} = & -(3\alpha\beta y |2x - \beta y^2|^{1/2} - 2\beta x + 2\beta^2 y^2 - y) w_{xx} \\ & - (3\alpha |2x - \beta y^2|^{1/2} + 2\beta y - \beta^{-1}) w_{xy}. \end{aligned}$$

After the change of variables $x = \frac{1}{2}(\tilde{x}^2 + \beta\tilde{y}^2)$, $y = \tilde{y}$, $w = \tilde{w}$ and dropping tildes we get

$$w_{yy} = \beta w_{xx} + \frac{1 - 3\alpha\beta x}{\beta x} w_{xy}.$$

This equation has a nontrivial symmetry $w_y - \lambda w$, $\lambda = \text{const.}$ The corresponding reduction $w = e^{\lambda y} v(x)$ yields ODE $v_{xx} = \lambda(3\alpha\beta x - 1)\beta^{-2}x^{-1}v_x + \lambda^2\beta^{-1}v$, which after the change of variables $v = \frac{\lambda}{2\beta}(3\alpha x - \ln x)\tilde{v}$ and dropping tildes takes the form

$$v_{xx} = \frac{\lambda}{4\beta^2} \left(\lambda(9\alpha^2 + 4\beta) - \frac{6\alpha\lambda}{\beta x} + \frac{\lambda - 2\beta}{\beta^2 x^2} \right) v. \quad (21)$$

Analysis of this equation splits into two branches. The first one corresponds to the case of $9\alpha^2 + 4\beta \neq 0$. In this case the scaling $\tilde{x} = \lambda\beta^{-1}(9\alpha^2 + 4\beta)^{1/2}x$ after dropping tildes gives Whittaker's equation

$$v_{xx} = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{\lambda(\lambda - 2\beta)}{4\beta^2 x^2} \right) v \quad (22)$$

with $\kappa = 3\alpha\lambda\beta^{-2}(9\alpha^2 + 4\beta)^{-1/2}$ and $\mu = \pm\frac{1}{2}(\lambda\beta^{-2} - 1)$. As it was shown in [16], this equation is integrable in quadratures whenever $\pm\kappa \pm \mu - \frac{1}{2} \in \mathbb{Z}$. Therefore for each choice of α and β there exists an infinite number of values for λ such that equation (22) is integrable in quadratures.

The second branch corresponds to the case of $9\alpha^2 + 4\beta = 0$. Then equation (21) acquires the form

$$v_{xx} = \left(\frac{A}{x} + \frac{B}{x^2} \right) v$$

with $A = \frac{27}{2} \alpha^3 \tilde{\lambda}^2$, $B = \tilde{\lambda}(\tilde{\lambda} - 1)$, and $\tilde{\lambda} = \frac{8}{81} \lambda \alpha^{-4}$. After the change of variables $v = \frac{1}{2} A^{-1/4} \tilde{x}^{1/2} \tilde{v}$, $x = \frac{1}{16} A^{-1} \tilde{x}^2$ and dropping tildes we have Bessel's equation

$$v_{xx} = \left(\frac{1}{4} + \frac{4B + \frac{3}{4}}{x^2} \right) v, \quad (23)$$

which is integrable in quadratures whenever $B = -\frac{3}{16} + \frac{1}{2} \left(n + \frac{1}{2}\right)^2$, $n \in \mathbb{Z}$, see [17]. So for each choice of α there exists an infinite number of values for λ such that equation (23) is integrable in quadratures.

4.3.4. Solution (16) with $\beta = 0$

The second equation of system (6) that corresponds to solution (16) in the case of $\beta = 0$ has the form

$$w_{yy} = (2\alpha x - 2\alpha^2 y^2 + y) w_{xx} - (2\alpha y - \alpha^{-1}) w_{xy}.$$

After the change of variables $x = \frac{1}{8} \alpha^{-3} (2\tilde{x} + \tilde{y}^2 + 2\tilde{y} - 1)$, $y = -\frac{1}{2} \alpha^{-2} \tilde{y}$, $w = \tilde{w}$ and dropping tildes we get

$$w_{yy} = 2(x + y) w_{xx} + w_x.$$

This equation has a nontrivial symmetry $w_x + w_y - \lambda w$, $\lambda = \text{const}$. The corresponding reduction $w = e^{\lambda x} v(\tau)$ with $\tau = x + y$ yields ODE $v_{\tau\tau} = -((4\lambda\tau + 1)v_\tau + \lambda(2\lambda\tau + 1)v)(2\tau - 1)^{-1}$, which after the change of variables $v = \tilde{x}^{-\lambda/2-1/4} e^{-\tilde{x}/2} \tilde{v}$, $\tau = \frac{1}{2}(\lambda^{-1}\tilde{x} + 1)$ and dropping tildes acquires the form of Whittaker's equation

$$v_{xx} = \left(\frac{1}{4} + \frac{3\lambda + 1}{4x} + \frac{(2\lambda + 1)^2}{8x^2} \right) v. \quad (24)$$

From results of [16] it follows that this equation is integrable in quadratures whenever $\lambda = 1 \pm 12n \pm (128n^2 - 32n + 6)^{1/2}$, $n \in \mathbb{Z}$. Therefore for each choice of α there exists an infinite number of values for λ such that equation (24) is integrable in quadratures.

4.3.5. Solution (17)

In the case of solution (17) we can put $\beta = 0$ without loss of generality. Then the second equation of system (6) takes the form

$$w_{yy} = \alpha w_{xx} - \alpha^{-1} x w_{xy}.$$

The nontrivial symmetry $w_y - \lambda w$, $\lambda = \text{const}$ leads to the reduction $w = e^{\lambda y} v(x)$, where v is a solution of ODE

$$v_{xx} = \lambda \alpha^2 x v_x + \lambda^2 \alpha v.$$

After the change of variables $v = e^{\frac{1}{4}\tilde{x}^2} \tilde{v}$, $x = \beta^{-1} \lambda^{-1/2} \tilde{x}$ and dropping tildes we have Weber's equation

$$U_{xx} = \left(\frac{1}{4} x^2 + \mu - \frac{1}{2} \right) U \quad (25)$$

with $\mu = \beta^{-1} \lambda^{-1/2}$, which is integrable in quadratures whenever $\mu \in \mathbb{Z}$, [11]. Therefore for each choice of α there exists an infinite number of values for λ such that equation (25) is integrable in quadratures.

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